Infinity in the regularization of Quantum Electrodynamics: a non-standard alternative

Jeffrey Bárcenas,* Luis Reyes–Galindo,† and Raúl Esquivel–Sirvent‡

Instituto de Física, Universidad Nacional Autónoma de México
Ciudad Universitaria, D. F. 01000, México.
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We review the concept of infinity as applied to regularization procedures in Quantum Electrodynamics. A clear distinction that is lacking in current literature is made between the physical contents of renormalization, and the mathematical aspects of regularization. Robinson's non-standard analysis is offered as a means to settle the ambiguities of the theory, in the spirit of Paul Dirac's well known comments concerning the weak status of the mathematics used in traditional regularization schemes. As a case study we consider the Casimir effect.

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"Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever".

Niels Henrik Abel

I. INTRODUCTION

Quantum Electrodynamics could arguably be called the most successful physical theory of the twentieth century. It has afforded extremely precise predictions of the most varied type of phenomena, at extreme energy ranges and scale lengths. And yet, even from the beginning it was subject to severe critics from many of its authors. We shall here concentrate upon a famous phrase by Paul Dirac when referring to the formal aspects of the theory: "I must say that I am very dissatisfied with the situation, because this so called good theory does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small - not neglecting it just because it is infinitely great and you do not want it!"

A. The mathematical concept of infinity

Infinity has always been one of the more esoteric ideas of Mathematics. What is infinity? Is it a number? Is it a concept? A limit? The question admits various answers, yet no answer is absolutely true in all cases. For example, in an entry-level Calculus college course one could ask what the limit is of the series

$$\lim_{N\to\infty}\sum_{n=1}^{N} n \tag{1}$$

The most correct answer, in the most of courses is *not* the immediate

$$\lim_{N\to\infty}\sum_{n=1}^{N} n=\infty \tag{2}$$

but rather *that the summation has no limit.* Yet this is not a canonical answer. One can change to the so called extended Real domain, and then say that the series' limit is infinity, or equivalently, diverges. One could even say that the concepts of "infinity" and of "divergence" are intertwined yet not equivalent. The problem can be complicated ad infinitum, for example in the complex plane, where one is actually forced to have the point at infinity as an actual entity, for example, as the poles of the Riemann sphere.

The point to emphasize is that mathematically, infinity is far from being a settled matter of discussion. So, what happens when infinity creeps up as an essential part of a Physical theory?

B. Renormalization and regularization

The year 1900 is remembered in Physics history as the birth date of Quantum Mechanics for the publication of M. Planck's famous solution to the black-body radiation spectrum problem. Less known is that about a decade later, Planck was forced to introduce an extra term to his formula, what is now known as the zero-point energy which arises in the energy spectrum of the generalized quantum harmonic oscillator. The addition of this term was a necessary step to make the theory consistent with Thermodynamics, and gave rise to Planck's Second Quantum Theory. The additional term, if taken at face value, gives a physically absurd result: that the total energy of the electromagnetic field is always a divergent quantity, i.e. the sum of the energy for all field modes diverges for any field that can be expressed as a set of harmonic oscillators.

*Electronic address: jeffrey@ciencias.unam.mx
†Electronic address: luisreyes@fisica.unam.mx
‡Electronic address: raul@fisica.unam.mx

This formal difficulty was passed over in silence by various arguments, such as the explanation that since physically relevant quantities are always associated to energy differences and not to total energy values, the divergent quantities could be swept away by a simple subtraction.

Still, infinity refused to stay out of the picture. The problem of the self energies had been a recurrent preoccupation of theoretical physics, and one that could simply not be circumvented by simplistic arguments when Dirac introduced the quantum theory of the electron that marked the beginnings of Quantum Electrodynamics. Particularly, two problems polarized opinions: the electron's self energy and the electron's coupling to the electric field. Although of immense historic interest, we shall not here deal with the development of the theory that led to the development of the mature theory that afforded Schwinger, Tomonaga and Feynman the Nobel Prize. Apart from the introduction of formal and conceptual frameworks in which to encompass the totality of electromagnetic phenomena in a manner fully compatible with quantum mechanics and special relativity, Quantum Electrodynamics introduced a novel concept in Physics as a whole: the concept of renormalization.

Renormalization is often wrongly conceived as the methodology of abstracting finite results from infinite quantities, yet this is incorrect. The mathematical methodology of obtaining finite results from the difference of infinite quantities is called regularization, and is only part of renormalization proper. Renormalization is much more ambitious and is more a matter of principles *and* methodologies applicable to a particular physical theory [25].

C. The cutoff

After renormalization was raised to a standard requirement of physical theories by F. Dyson's work on the mathematical unity of the Schwinger-Tomonaga-Feynman theories, a new conceptualization was brought by the introduction of "effective field theories", following an influential article by Lewis, where we find the key phrase [25]

"The electromagnetic mass of the electron is a small effect and that its apparent divergence arises from a failure of present day quantum electrodynamics above certain frequencies"

This marked the appearance of a new conception of physical law, that absolutely departed from the foundationalist attitude of the first half of the twentieth century (although one could argue that it an essential extension of orthodox quantum mechanics). In this new paradigm, Physics was no longer concerned with the search of " final" theories, but only with empirically successful approximations with predefined and limited validity. Therefore, when one formulates a theory, QED for example, one must admit the theory's failure above a certain energy range. This upper limit is known as the cutoff. When carrying out calculations, for example the sum over all energy eigenmodes of the electromagnetic field to find the total energy of a system, it is understood that there is always a real valued limit after which all eigenvalues are zero. The reason often cited for this approach is that one expects the theory to fail after the cutoff, but that it is not unreasonable to expect that only the lower energy ranges determine the behavior of the system below the cutoff.

Yet there are problems, deep problems for this approach. There is no proof that QED *must* fail after a given energy range, however likely or not this be. And *if* it does fail, there is no way to determine what this cutoff is beforehand. So how does one introduces an actual cutoff? This is probably the most insensible bit of all, as we shall see in the calculation of Casimir energies. After an "ultraviolet" cutoff is introduced to begin the calculation, a physically meaningful expression is arrived at which depends on the cutoff. Yet the very final step is to take the cutoff to infinity. We hope the reader can appreciate the circularity. So it seems that either the cutoff is ontologically significant, and therefore the matter of taking it to infinity in the end contradicts the principles of effective field theories, or it is insignificant, but then the actual calculations find little justification.

Dirac suspected that the problems were not only physical, but mathematical. The fact is that our standard mathematics are not successful at incorporating infinity at large.

De la Peña has shown that in the framework of stochastic electrodynamics (SED) there are solid arguments that point to the necessity of a cutoff function as a true physical requirement [13]. He argues that the introduction of a cutoff is equivalent to the introduction of a structure to, for example, the electron as a particle. In SED the cutoff is also a requirement to keep causality intact in high energy ranges and to "recover consistency of the description".

As an example of recent uses of cut-off in the calculation of Casimir eneries, Edery [26] developed a multidimensional cut-off technique to calculate the Casimir energy for massless scalar field in d-dimensional rectangular cavities.

Therefore, one must not conclude that cutoffs are always dispensable. What we blatantly oppose is the prescription of a cutoff as a physical requirement when it is actually a mathematical one. One must carefully distinguish such requirements, as we hope the above analysis has shown.

II. NONSTANDARD ANALYSIS: THE MATHEMATICS OF THE INFINITE

Nonstandard Analysis (NSA) is a relatively new mathematical discipline begun in 1961 by Abraham Robinson [2]. NSA works upon an extension of the real numbers

by including "new" entities: infinitesimally small and infinitely large numbers. The quotation marks are included because any physicist has at some time or another been exposed to the use of infinitesimals, e.g. the common use of "infinitesimal notation" such as dW to signify an "infinitesimal amount of work", the method of "virtual work", etc. What is often referred to as physicists' sloppiness was actually a fruitful method of proof for the likes of Leibnitz, Newton and Euler, but that was shunned by later generations of mathematicians due to its lack of rigorous foundation [4, 5] . One of the most attractive uses that NSA has yielded is the formal mathematical justification of the use of these infinitesimals. Several textbooks have now been published that use infinitesimals as the formal grounds upon which to build the entire calculus, gaining the advantage of having much shorter and extremely intuitive proofs over the classic epsilon/delta formulation [7]. Since its inception, A. Robinson and K. Gödel were convinced that not only would NSA be an extremely economic shorthand notation for constructing new compact proofs of old theorems (which it has!), but that it would also become the basis for the search of ultimately new mathematical statements, practically and even factually unprovable in Standard Analysis [2]. As initially formulated by Robinson, NSA required at the very least a good acquaintance with the principles of formal logic, a fact which turned away many a mathematician, and made the field of research quite limited despite the abundance of possibilities it offered.

Most of the work done with NSA has centered upon the *infinitesimal* part, with applications in differential geometry, statistics and various other mathematical branches. However, little attention has been given to the *infinite* segment of the so-called hyperreals. The rest of this article will be devoted to show that Theoretical Physics, Quantum Field Theory (QFT) in particular, may find it fruitful to look at NSA as a new set of tools to clarify long standing controversies or loopholes in its formal and even ontological repository of knowledge.

A. A short introduction to hyperreal entities

Following Robinson's seminal work, alternative and equivalent axiomatic formulations of NSA were created, in an effort to simplify the conceptual framework and limit to a minimum the prerequisites of mathematical logic necessary to introduce NSA to new audiences. Amongst them, we might mention Nelson's Internal Set Theory –actually an extension of Zermelo-Frankel set theory– as particularly accessible. However, we will base our efforts on the so-called ultrapower construction, as it offers an immediate application for our present purposes [3]. We will make no effort to introduce our reader to the formal construction of the hyperreal numbers, as this can be consulted in the referred works [2, 3, 5]. Our purpose here is to illustrate the use of new mathematical entities and to briefly state some of their more interesting properties, and how they are related to well known practical, mathematical and philosophical problems of Quantum Field Theory.

In all its formulations, NSA can be viewed as an "enlargement" of the classic analysis familiar to theoretical physicists. This enlargement is carried out by postulating new entities (e.g. in addition to the 'standard' real numbers of old, the logical possibility of new 'nonstandard' elements is postulated) and additional axioms are appended to the old axiomatic set. This last step is crucial, since the old axioms are not changed and thus all arithmetical properties for the standard numbers remain valid for the new nonstandard elements, as well as *some* specific relations between sets made up of these elements, and between sets themselves. The new axioms in part serve to specify which of these relations remain valid for the nonstandard elements, and how a strictly nonstandard relation may be scrutinized to give out standard results. This procedure is a particular case of a method that can be carried out with any mathematical language, and that stem's from the work of K. Gödel. Thus, Real Analysis can be extended to Nonstandard Hyperreal Analysis. Zermelo-Frankel set theory can be extended to Internal Set Theory with the addition of Nelson's postulates and axioms.

The new entities introduced by NSA in addition to the common real (henceforward 'standard') numbers are the nonstandard numbers, further divided into two classes, infinitesimal and unlimited. Since nonstandard numbers inherit real number arithmetic, any well defined operation (well defined in accordance to the newly added transfer axioms) can be manipulated by established rules and algorithms.

In the ultrafilter characterization of Nonstandard Analysis (NSA) any number both standard and nonstandard can be constructed as an equivalence class of infinite series of real numbers. A standard number α with positive absolute value x, for example, is characterized by the real valued constant series given by

$$\alpha = \{x, x, x \ldots\}. \tag{3}$$

The characterization is not unique. For example, the series

$$\alpha_1 = \{0, 0, 0, x, x, x \ldots\}, \tag{4}$$

may represent the same number. The ultrafilter construction of the equivalence relation is such that series which have a large "coincidence set" in their entries are considered equal, and hence both of the above series represent the same number, as their coincidence set is 'large' (a property which can be defined in a strict and formal sense using ultrafilters through the "almost everywhwere" condition familiar to topologists).

For, our purposes, it is enough to point out that all series ultimately represent a hyperreal: series whose limit

tends to zero are equivalent to infinitesimal hyperreals, while unbounded series are equivalent to unlimited hyperreals. These latter types are all grouped under the same name in Standard Analysis. Let us take as example an obviously divergent series,

$$a_n = \left\{ \sum_{n=1}^{N} \right\} = \left\{ 1, 3, 6, \ldots, \sum_{n=1}^{N} \right\}. \tag{5}$$

To signify the divergence at "infinity", in Standard Analysis, one uses the notation

$$\lim_{n \to \infty} a_n = \infty. \tag{6}$$

The ∞ symbol is nothing but a shorthand notation of this fact, and in no way signifies that the limit of the sums *is* a well defined number ∞. In fact, the series *has no limit*. Usage of the extended field $\{\mathbb{R} \cup \infty\}$ does not solve the problem either.
Now consider the following series,

$$b_n = \left\{ \int_0^N dx \right\} = \{1, 2, 3, \ldots, N\}, \tag{7}$$

obviously divergent as well. Comparing entry by entry, we might be tempted to think that a_n is 'larger' than b_n, since each entry of the first is larger than each entry of b_n. What about comparing the limits when the series tend to infinity?

$$\lim_{N \to \infty} a_n = \infty, \qquad \lim_{N \to \infty} b_n = \infty. \tag{8}$$

In the strictest sense, Standard Analysis has no formal way to acknowledge this difference. NSA on the other hand, has much more to say. We could ask ourselves, for example, what the difference between the numbers represented by the infinite series a_n and b_n is. Since subtraction is a well defined operation of hyperreal numbers, it is a legitimate question. Subtraction is defined term by term (analogous to vector subtraction),

$$\begin{aligned} &\{a_n\} - \{b_n\} = \{a_1, a_2, a_3, \ldots\} - \{b_1, b_2, b_3, \ldots\} \\ &= \{a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots\} = \{c_1, c_2, c_3, \ldots\} \\ &= \{c_n\}, \end{aligned} \tag{9}$$

where $\{c_n\}$ now represents another hyperreal, which in general can be both limited or unlimited, standard or nonstandard. The question concerning the difference between two standard formally undefined (divergent) numbers, which is naught but a nonsensical question in the old framework, becomes a fully legitimate question in NSA [2, 5].

As mentioned, hyperreal numbers are classified into several groups, where ${}^*\mathbb{R}$ denotes the hyperreal set (the $^*-$ notation is commonly used to denote nonstandard entities in contrast to standard ones).

Definition 2.1 A hyperreal number b is:

- *limited* if $r < b < s$ for some r,$s \in \mathbb{R}$.
- *positive unlimited* if $r < b$ for all $r \in \mathbb{R}$.
- *negative unlimited* if $b < r$ for all $r \in \mathbb{R}$.
- Unlimited if it is positive or negative unlimited.
- *positive infinitesimal* if $0 < b < r$ for all positive r $\in \mathbb{R}$.
- *negative infinitesimal* if $r < b < 0$ for all negative $r \in \mathbb{R}$.
- *appreciable* if it is limited but not infinitesimal, i.e., $r < |b| < s$ for for some r, $s \in \mathbb{R}$.
- *multiplicative inverses.* Any number of the form $1/\epsilon$ is unlimited when ϵ is infinitesimal, *mutatis mutandis* for the inverse of an unlimited number.

The following definitions are conveniently formulated in more modern texts.

Definition 2.2 A Hyperreal b in *infinitely close* to a hyperreal c, denoted by $b \simeq c$, if $(b - c)$ is infinitesimal. This defines an equivalence relation on ${}^*\mathbb{R}$, and the *halo* of b is the $\simeq$-equivalence class

$$\mathbf{hal}(b) = \{c \in {}^*\mathbb{R} : b \simeq c\}.$$

Definition 2.3 Hyperreal numbers b and c are *limited distance apart*, denoted by $b \sim c$ if $(b - c)$ is limited.The *Galaxy* of b is the $\sim$-equivalence class

$$\mathbf{gal}(b) = \{c \in {}^*\mathbb{R} : b \sim c\}.$$

Theorem 2.1 *Every limited hyperreal b is infinitely close to exactly one real number, called the **shadow** of b, denoted by* $\mathbf{sh}(b)$. This leads to the fact that the ${}^*\mathbb{R}$ is denser than $\mathbb{R}$!

B. Axiomatic development of Nonstandard Analysis

Once the hyperreals are constructed, one may chose whichever axiomatic formulation of NSA is desired to work in the extended system. In all formulations, of particular importance are the *transfer* criteria, that is, the rules that regulate which relations that are known to hold for standard numbers also hold for arbitrary hyperreals. Although hyperreals follow standard arithmetic, one

must be careful, for example, when intending to transfer properties between sets. Although a complete understanding of the transfer principle requires acquaintance with formal logic, it can be loosely stated as follows.

Universal transfer Principle: *if a property holds for all real numbers, then it holds for all hyperreal numbers.*

Existential transfer Principle: *if there exist a hyperreal number satisfying a certain property, then there exist a real number with this property.*

III. CASE STUDY: THE CASIMIR EFFECT

The Casimir effect, proposed by its namesake in 1948 [11], has often been referred to as "one of the least intuitive results in Quantum Field Theory"[13, 14, 16]. Two neutral parallel plates separated a distance L attract each other with a force proportional to the inverse fourth power of the separation. There are several reasons for this assessment. First of all, as postulated by Casimir [11], the force arises because of the disruption of the "zero-point vacuum electromagnetic field" by material body boundaries, and many a discriminating reader will heartily point out that the term in quotations is not an overly intuitive notion. An alternative interpretation due to Lifschitz [12] proposes that the zero-point electromagnetic field oscillations polarize the material bodies' boundary molecules, and that the force is due to the interaction of these molecules.

Despite all this, the Casimir effect is one of the most trivial examples where infinities are "subtracted", and it serves well to illustrate the nonstandard analysis approach.

In essence, the classic approach is to subtract the energy density of the parallel plate configuration from the energy density when the separation of the plates goes to infinity, and to take the usual derivative for calculating the force per unit area. The difficulties arise because, both energy densities are strictly infinite and classical mathematics has no justification for the said operations.

A particularly troubling issue has plagued Casimir's original conception of the effect that bears his name [11]. It is the following question : "What is infinity minus infinity?" Although the answer to this question can be obtained from physical arguments as done by Casimir, it is nonsensical in the context of standard analysis [9].

Casimir found a way to answer the question. Since his time, others have found other means to arrive at the same answer [12, 13, 14, 16, 23, 24], using similar methods, the above mentioned cutoff or regularization procedures. However, the ontological ("physical") question of what these divergences really mean remains unanswered, in addition to several other matters which we will here point out as unsatisfactorily settled.

The fact remains that in the standard viewpoint, "infinity minus infinity" is simply not a well formed question, despite that the question was answered. A close look at the regularization procedure and the argumentation for the cutoff function shed light into the subject, and for years physicists have actually been using poorly justified mathematics to obtain brilliant results. It is now our task to give firm formal foundation to the cutoff procedure and to justify in a rigorous manner Casimir's result. Furthermore, the effect is not only a esoteric blackboard result, lurking into the theoreticians' minds but also an experimental fact stated by several groups around the world [17, 18, 19, 20]. The effect, as the theory predicts, is a quantum result with no equivalent in the classical world. In spite of several problems on the development of the experimental set-up, the problem remains the same: we get a strange theoretical result, stated with redundant mathematics, explained with clever physical arguments and with a quantum reality supported by measurable facts. As we see, the problem is not too obvious to justify hiding it under the "desk rug".

A. On cutoff independence

Cutoff independence can be defined in two ways. The first is to say that in deriving a physical result, no actual mention of a cutoff should be made. Yet without an actual cutoff scheme, obtaining actual results is impossible, since we do not know how to deal with actual infinite quantities. Schwinger's approach is a modified view of this viewpoint, denying the reality of the cutoff, but obtaining physical results by the introduction of a finite number of parameters that depend on experimental data. Still, if one accepts renormalization as a theoretical necessity, a non-realistic interpretation of the cutoff is inadmissible. Furthermore, this ultimately pragmatic approach abandons the hope of physical justification, as Dirac pointed out. A second option is the admission of a physically significant cutoff in the procedure (a realist interpretation), to which must be added a proof that the end result does not depend on the chosen cutoff parameter. In the standard framework, this leads to a well known paradox on the reality of the cutoff and the interpretation of the physical significance of the theory. [25]. We would like to keep the significance of the cutoff, yet avoid the paradox. We will therefore adopt a variation on the second outlook by introducing an actual cutoff in the calculations, stated a priori, yet we will obtain the same results for every cutoff used. To do this, we shall have to adopt a new class of numbers, a new class of mathematical entities. Yet this class of numbers has the strongest logico-mathematical foundations behind it. As we shall see, this approach is in essence completely unlike the standard and orthodox one, which loses all information and significance concerning the cutoff by sending it to infinity at the end of the calculations.

B. The original Casimir effect in the NSA framework

We will follow Casimir's argument almost verbatim [11]. The energy (per unit of area) of the Casimir system E is defined as the difference between the energies of the parallel plate configuration E_p and of free space E_v,

$$E = E_p - E_v, \tag{10}$$

where the energies are given by the integrals over momentum space

$$E_p = \frac{\hbar c}{2}\frac{L^2}{(2\pi)^2}\int_0^\infty\int_0^\infty dk_x dk_y \left(\sqrt{k_x^2+k_y^2} + 2\sum_{n=1}^{\infty}\sqrt{k_x^2+k_y^2+\left(\frac{n\pi}{L}\right)^2}\right), \tag{11}$$

and

$$E_v = \frac{\hbar c}{2}\frac{L^2 d}{(2\pi)^3}\int_0^\infty d\vec{\mathbf{k}}\; 2\sqrt{k_x^2+k_y^2+k_z^2}\;, \tag{12}$$

the expression in the eq. (11) and (12) can be further simplified using the substitution

$$k_\perp = \sqrt{k_x^2+k_y^2}, \tag{13}$$

using the eq.(10) and integrating over the area element $dk_x dk_y = 2\pi k_\perp dk_\perp$ we obtain

$$E = \int_0^\infty dk_\parallel\; k_\parallel \left(k_\parallel + 2\sum_{n=1}^{\infty}\sqrt{k_\parallel^2+\left(\frac{n\pi}{L}\right)^2}\right) - \left(\frac{2L}{2\pi}\int_0^\infty dk_z\sqrt{k_\parallel^2+k_z^2}\right). \tag{14}$$

We call attention to the following fact: as they stand, the eq. (14) for E_p and E_v are divergent, i.e. infinite. Casimir further defined the "change of variable" $k_z = n\pi/L$, where n is viewed as a continuous variable. Although n was previously assigned to a discrete variable, let us admit this step to allow a coupling for formal purposes and using the visual simplification $k_\perp = z$ end up with the expression

$$E(L) = \frac{\hbar c}{2\pi}\frac{\pi}{L}^3\int_0^\infty dz\; z \left[\frac{z}{2} + \sum_{n=1}^{\infty}\sqrt{z^2+n^2} - \int_0^\infty dn\sqrt{z^2+n^2}\right]. \tag{15}$$

The last step before the actual regularization begins is to define the function

$$E(n) \equiv \int_0^\infty dz\; z\sqrt{z^2+n^2}, \tag{16}$$

which is of course also infinite. Casimir circumvented this fact by appealing to the following argument. Even though the plates are hypothesized as perfectly conducting, hence perfect reflectors for all frequencies, "real" plates actually become transparent to high frequency photons. This is an empirical fact. Casimir proposes to include this fact in the calculation by introducing a cutoff-function $f(w)$ which is nothing but the Heaviside step-function translated to the cutoff frequency λ. After introducing the cutoff, Casimir used the Euler-Maclaurin summation/integration formula to obtain a finite result.

Standard Analysis as taught in most college courses would find the above derivation unacceptable on several grounds. In particular, the "cutoff-function" step is completely out of bounds, as one cannot appeal to a "physical notion" (however justifiable [13, 14, 16]) in the deduction of a strictly mathematical result. A formal mathematical alternative is then called for. NSA offers an immediate answer. In this framework, the subtraction of two infinite quantities is no different from the subtraction of two finite real numbers in standard analysis. In the following paragraphs, Casimir's result will arise from the use of hyperreal valued functions (an extension of the definition of a hyperreal number using the series approach). The important fact is that to state the *existence* of the subtraction given by $E = E_p - E_v$ no mention will be made of a cutoff function.

Our first goal is the Euler-Maclaurin formula (EM) which is an approximation by series expansion to the integral of a continuous function between arbitrary integrals [10]. The first question is if there exist a extended version in the hyperreal domain. If it so, the Casimir energy and the cutoff-free calculations would make the Casimir energy and force perfect, with no "high frequency" arguments for real materials. The following theorem proves the existence of the extended EM formula

Theorem 5.1. *The Euler-Maclaurin Formula in the standard domain is well–defined in the hyperreal domain.*

Proof Let $f(x)$ be continuous of degree $2n$.Its integral between the interval (a,b) of length $h=(b-a)/m$ is given by the n-order approximation

$$\begin{aligned}\sum_{k=0}^{m} f(a+kh) = \qquad\qquad\qquad\qquad (17)\\ \frac{1}{h}\int_a^b f(t)dt + \frac{1}{2}\Big\{f(b)-f(a)\Big\}\\ +\frac{1}{2}\sum_{k=1}^{n-1}\Big(\frac{h}{2}\Big)^{2k-1} B_2k\Big\{f^{2k-1}(b)-f^{2k-1}(a)\Big\}\\ -\frac{h^{2n}}{(2n)!}B_{2n}\sum_{k=0}^{m-1} f^{2n}(a+kh+\theta h),\end{aligned}$$

where $0<\theta<1$, and B_n is the nth Bernoulli number. Notice for a fixed a the EM formula can be stated in the form "for every $b\in\mathbb{R}$, then ..." and hence is subject to transfer. Then, the EM formula is valid for arbitrary hyperreal integrations limits a and b. $\square$

For the Casimir problem, $a=0$ and $h=1$ and $f(n)=E(n)$. Eq. (17) then reads

$$\begin{aligned}\sum_{k=0}^{m} f(k) = \int_0^m f(t)dt + \frac{1}{2}\Big\{f(m)-f(0)\Big\} \qquad (18)\\ +\frac{1}{2}\sum_{k=1}^{n-1}\Big(\frac{1}{2}\Big)^{2k-1} B_2k\Big\{f^{2k-1}(m)-f^{2k-1}(0)\Big\} - R(m),\end{aligned}$$

where R is the remainder term given by

$$R(m) = -\frac{1}{(2n)!}B_{2n}\sum_{k=0}^{m-1} f^{2n}(k+\theta). \qquad (19)$$

In the final expression, only the second term differs from zero in the summation on the right hand side of equation (18). The function $f(x)$ is C^∞, that is, the derivative of *any* arbitrarily high order exists (derivatives of unlimited order included), the referred summation always being strictly zero. Notice that the remainder term, eq. (19) is proportional to $1/(2n)!$, and thus for an unlimited integer (see definition 2.1) is actually infinitesimal. To prove the last assertion, we need the following lemma

Lemma 5.1 *In the Euler-MacLaurin (EM) formula extended to the hyperreal domain,the remainder is infinitesimal.*

Proof All higher order derivatives $f^{(n)}(0)=0$, whenever $n>3$ for the Casimir Problem, in the hyperreal domain. By our previous definitions, $R(m)$,$R(n)\simeq 0$. $\square$

The remainder is not unique, but is always infinitesimal for an unlimited upper integration/summation limit. Yet, one would expect an infinitesimal quantity to be physically unmeasurable, and so infinitesimal differences amount to strictly equal physical measurements. The final expression for the Casimir problem reads

$$\begin{aligned}\sum_{k=0}^{m} f(k) = \int_0^m f(t)dt+\frac{1}{2}\Big\{f(m)-f(0)\Big\} \qquad (20)\\ +\frac{f^{(3)}(0)}{30\cdot 4!}+\zeta_{inf}(m).\end{aligned}$$

Armed with this results, the Casimir Energy can be calculated immediately.

Corollary 5.1 *The Casimir energy per unit area is given by*

$$\begin{aligned}E(L) = \frac{\hbar c}{2\pi}\left(\frac{\pi}{L}\right)^3\Big\{f(0)+\alpha(\lambda)-\beta(\lambda)\ \Big\}+\zeta_{inf} =\\ =\eta\frac{\hbar c}{2\pi}\left(\frac{\pi}{L}\right)^3+\zeta_{inf},\end{aligned}$$

where η is the hyperreal constant obtained when evaluating the given composite function, restricted to an unlimited λ. This energy is perfectly and unambiguously defined.

Proof In analogy to the definition of a hyperreal number, we define the hyperreal function given by the series

$$f_m(n) = \left\{\ \int_0^m dz\ z\sqrt{z^2+n^2}\ \right\} \equiv {}^*f(n).$$

As a first step, the integral in eq. (15) becomes

$$E(L) = \frac{\hbar c}{2\pi}\left(\frac{\pi}{L}\right)^3\left[\frac{{}^*f(0)}{2}+\sum_{k=1}^{\lambda}{}^*f(k)-\int_0^\lambda dn\ {}^*f(n)\right], \qquad (21)$$

where λ is a standard entity. We extend it to the hyperreal domain using the composite hyperreal functions

$$\alpha(\lambda) = \left\{\int_1^\lambda dn\ f(n)\ \right\},$$

and

$$\beta(\lambda) = \left\{ \sum_{k=0}^{\lambda} f(k) \right\}.$$

Where λ is in general any hyperreal number. Instead of taking the limit to infinity *a posteriori*, it is only necessary to define λ as an unlimited number. We then insert the composite hyperreal valued functions in eq.(21), and use the equation (20) from which the EM formula stems. We immediately transfer into NSA language,

$$\sum_{k=0}^{*\lambda} f(k) = \int_0^{*\lambda} f(t)dt + \frac{1}{2}\Big\{f(*\lambda) - f(0)\Big\} + \frac{f^{(3)}(0)}{30 \cdot 4!} + \zeta_{inf}(*\lambda). \quad (22)$$

When inserted into the final expression for $E(L)$, eq. (15) one finds by the theorem 5.1, that only the second term is different from zero

$$f(0)+\alpha(*\lambda)-\beta(*\lambda) = \frac{1}{2}f(*\lambda)+\frac{f^{(3)}(0)}{720}+\zeta_{inf}(*\lambda), \quad (23)$$

Finally, the energy can be expressed as

$$E_\lambda(L) = \frac{\hbar c}{2\pi}\left(\frac{\pi}{L}\right)^3 \left\{ f(0) + \alpha(*\lambda) - \beta(*\lambda) \right\} = \eta\frac{\hbar c}{2\pi}\left(\frac{\pi}{L}\right)^3 + \zeta_{inf}(*\lambda), \quad (24)$$

where

$$\eta = \frac{1}{720},$$

is a hyperreal constant. $\Box$

Remark The equation (24) is Casimir's result. Since it is valid for arbitrary unlimited $*\lambda$, a different choice of upper integration limit yields the same result, but differing in an infinitesimal amount or in modern notation

$$E_{*\lambda}(L) - E_{*\lambda'}(L) \simeq 0,$$

therefore

$$\mathbf{sh}(E_\lambda(L)) = \mathbf{sh}(E_{\lambda'}(L))$$

for any two unlimited hyperreals. Notice also that any function which multiplies the integrand that is infinitesimally close to unity near the zero frequency and infinitesimally valued in the unlimited domain gives the same result. This is in accordance to the requisite behavior of a classic cutoff function (see [14]). Hence, for a cutoff of this type, the result is independent of the cutoff function for any unlimited upper limit. This is of course true for the classic exponential function that gives rise to the zeta function regularization method [21, 22].

Corollary 5.2 *The Casimir force in the standard framework is cutoff independent if and only if the cutoff function differs in an infinitesimal amount from unity when valued at any unlimited number. Its value only depends on fundamental constants and the separation between the plates. In other words, the force per unit of area is*

$$F_C = -\frac{\pi^2 \hbar c}{240\ L^4}.$$

Proof follows from the derivative of the energy given by the corollary 5.1, and from the above remark.

$$F_C = -\frac{\partial E}{\partial L}. \qquad \Box$$

IV. FINAL REMARKS

We have shown how in the NSA framework, the Casimir effect as originally postulated is a perfectly defined problem, with a perfectly defined answer. Let us recount our steps. We first defined the energy of the vacuum and that of the plates in Nonstandard language, and showed that they lead to well defined hyperreal function; that is, we proved existence. We then proved, using the hyperreal extension of the Euler-Maclaurin formula, that the subtraction is unique up to an infinitesimal amount whenever the energy is calculated for an unlimited hyperreal, but argumented that when one considers only the shadow of the resulting quantity, the result is always the same; hence, we proved uniqueness in the real standard domain. Since the choice of a particular cutoff yields the same result. when the cutoff belongs to the unlimited domain, the result is cutoff independent.

One should not fear the notion of infinity and its manipulation, but the manipulations should be done in a clear and logical coherent manner, as Dirac justly desired.Although QFT has been one of the most fruitful and exact field of contemporary physics, it is plagued all over by these sort of infinite quantities. Indeed, most textbooks on the subject insist in reminding students of QFT from the start that as with all physical theories, QFT must one day find its limitations, probably in the range of very high energies where infinities abound. We offer an alternative picture and insist on the following two facts:

a) That QFT may or may not be a 'final theory' is an epistemological question that will remain unanswered until one finds a better or alternate theory, or when practical problems in Physics finds enough cause to require a better explanation [23]. Until then, QFT remains physicists flagship as the most accurate theory known to date. Furthermore, its limitations ought not to be implied in

its mathematical *range*, since NSA affords exact computations for any known range of energies (i.e. the plates become invisible to high frequencies waves)

b) The requirement of a cutoff function should *not* be stated as a physical requirement in the original Casimir effect. It is a mathematical requirement given our current incomplete understanding of the operation upon unlimited hyperreals, but even then can be avoided if the problem is phrased in Nonstandard language. Usage of NSA affords two clear advantages: one can produce existence theorems for actual results, and can easily supply well defined methods for obtaining physically valid results in the real standard domain.

Some of the aforementioned limitations are due to the lack of work in the field of NSA. The theory guarantees that there are logically true conjectures of standard mathematics that are factually unprovable in standard terms and yet provable through nonstandard means. What is lacking is a powerful NSA-pure relation equivalent to, say the Cauchy integral theorem in Complex Analysis. However, with or without such results NSA offers a most solid ground upon which to heuristically understand, explain and calculate without ambivalence some of the most surprising physical results of the past century.

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